Int. J. Heat Mass Transfer. Vol. 10, pp. 541–551. Pergamon Press Ltd. 1967. Printed in Great Britain

HEAT TRANSFER IN THE THERMAL ENTRANCE REGION OF CIRCULAR TUBES AND ANNULAR PASSAGES WITH FULLY DEVELOPED LAMINAR FLOW

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(*Received* 15 *July* 1966)

Abstract—Series solutions of the type first proposed by Mercer [9, 10] are presented for laminar flow in tubes and annular passages (including the plane duct) with a step-change in either the temperature or the heat flux at one wall. This type of solution—which may be regarded as an extension of the well-known Lévêque-solution [8]—constitutes a convenient alternative to the eigenvalue-solutions in the first part of the thermal entrance region where a large number of terms is required in the eigenfunction expansions.

NOMENCLATURE

- A, non-dimensional velocity gradient at the wall;
- a_n, coefficients in expansion of velocity profile;
- B, $(1 - r^*)/(-\ln r^*)$;
- D_h, hydraulic diameter;
- M, $1 + r^{*2} - B$;
- Nu, Nusselt number, $q''D_h/[k(T_w - T_m)]$;
- p, parameter
  - 1 for solutions of the second kind;
  - 0 for solutions of the third kind;
- Pe, Péclet number, $u_m D_h/\alpha$;
- q'', heat flux at the wall;
- r, radial co-ordinate;
- $\bar{r}$, r/r_o;
- r^*, r_i/r_o;
- R, $s(1 - r^*)/\bar{r}_j$;
- s, parameter $\begin{cases} +1 \text{ for subscript } j = i; \\ -1 \text{ for subscript } j = o; \end{cases}$
- T, temperature;
- $\bar{T}^{(2)}$, $(T - T_e)/[(r_o - r_i)q''/k]$;
- $\bar{T}^{(3)}$, $(T - T_e)/(T_w - T_e)$;
- u, axial velocity;
- $\bar{u}$, u/u_m;
- x, axial co-ordinate;
- $\bar{x}$, $4(x/D_h)/Pe$;
- $\bar{y}$, $s(\bar{r} - \bar{r}_j)/(1 - r^*)$.

Greek symbols

- α, thermal diffusivity;
- η, similarity variable, $\bar{y}/\xi$;
- $\theta^{(2)}$, $\bar{T}^{(2)}/\xi$;
- $\theta^{(3)}$, $\bar{T}^{(3)}$;
- ξ, $(9\bar{x}/A)^{\frac{1}{3}}$;
- Φ, $q''D_h/[k(T_w - T_e)]$.

Subscripts

- e, entrance;
- i, inner wall;
- j, heated wall;
- k, opposite wall;
- m, mean value;
- o, outer wall;
- $x, r, \bar{x}, \bar{y}$, partial differentiation with respect to the indicated variable.

Superscripts

- (2), solution of the second kind;
- (3), solution of the third kind;
- $'$, differentiation with respect to η.

INTRODUCTION

THE PROBLEM of heat transfer with hydrodynamically developed laminar flow in tubes and other, geometrically simple, passages has traditionally been solved by the technique of

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separation of variables, that is, as an eigenvalue problem. For the classical Graetz problem, the circular tube and the flat duct with a step-change in wall temperature, extremely accurate values for the first eleven (ten for the flat duct) eigenvalues and eigen-functions have been computed by Brown [1]. The tabulation of the eigenfunctions has subsequently been extended up to order fifteen by Larkin [2] who used the asymptotic eigenvalues found by Sellars, Tribus, and Klein [3]. Siegel, Sparrow and Hallman [4] presented a direct solution to the case of the circular tube with prescribed heat flux and computed the first seven eigenvalues and the related constants. For the annular duct with a heated core and adiabatic outer wall Hatton and Quarmby [5] have given the first ten eigenvalues and related constants corresponding to a step-change in both the wall temperature and the heat flux at the wall. The computations were carried out for a range of radius ratios, including the limiting case of the flat duct. A very thorough treatment of the annulus problem has been published by Lundberg, McCuen and Reynolds [6], based on four types of fundamental solutions [7]—a step-change in the wall temperature or in the heat flux at one wall combined with either zero heat flux at the opposite wall or the opposite wall kept at the inlet temperature of the fluid.

The eigenvalue method yields an exact solution for the entire thermal entrance region but, near the point of a step-change in either the wall temperature or the heat flux at the wall, a large number of terms in the eigenfunction expansion is required. For this reason the similarity solution due to Lévêque [8] is commonly used immediately downstream of the step-change. However, since this approximation is valid only in a very restricted range, there is a need for a simple solution bridging the gap between the limit for the Lévêque solution and the point where the eigenvalue solution becomes manageable. Such a solution has, in fact, been proposed by Mercer who presented a series solution of the boundary-layer type for the flat duct [9] and the circular tube [10] with a step-change in the wall temperature. Apparently, however, Mercer did not recognize the significance of this method for computing heat-transfer parameters in the upstream part of the thermal entrance region, and he computed only the temperature profiles.

The present author arrived at this type of solution, for a wider class of problems, from a slightly different angle, namely by considering a perturbation of the Lévêque solution. The series solution obtained in this way allows the relaxation of the two most severe restrictions of the similarity solution: that the velocity profile be linear throughout the depth of penetration of the temperature signal; and that the effect of curvature is neglected. Two remaining limitations which indicate the boundary-layer aspect of the solution and cannot be removed by this technique are: (i) that the range of the similarity variable goes to infinity, and (ii) that far away from the wall the temperature approaches the inlet temperature of the fluid. These two conditions will, ultimately, limit the range in which the solutions are applicable but, as the numerical results will show, they are not quite so restrictive as one might be inclined to think.

Since condition (ii) also implies that the temperature gradient approaches zero far away from the wall, it follows that this type of solution does not allow a distinction between different kinds of boundary conditions at the opposite wall. Beyond the point where there is a significant difference between the two types of fundamental solutions, with the opposite wall either adiabatic or kept at the inlet temperature, the present solutions will correspond to the adiabatic condition.

SOLUTION OF THE ENERGY EQUATION

In the formulation of the problem we shall assume that the fluid properties are constant and that both the viscous dissipation and axial conduction are negligible. For fully developed flow in a duct with cylindrical symmetry the energy equation then becomes

$$u(r)T_x = \frac{\alpha}{r}(rT_r)_r \quad (1)$$

with

$$T(0, r) = T_e, \qquad r_i \leqslant r \leqslant r_o. \quad (2)$$

The remaining boundary conditions for the four fundamental solutions defined by Reynolds, Lundberg and McCuen [7] are stated below. Here and in the following the subscript j refers to the heated wall and k to the opposite wall while the subscripts i and o refer to the inner and outer wall, respectively.

$$j, k = i, o$$

and

$$s = \begin{cases} +1 & \text{for } j = i \\ -1 & \text{for } j = o. \end{cases}$$

Solutions of the first kind. (Temperature step at one wall; the opposite wall kept at the inlet temperature):

$$\left.\begin{aligned} T(r_j, x) &= T_w \\ T(r_k, x) &= T_e \end{aligned}\right\} \quad x > 0. \quad (3)$$

Solutions of the second kind. (Step in heat flux at one wall; the opposite wall adiabatic):

$$\left.\begin{aligned} T_r(r_j, x) &= sq_j''/k \\ T_r(r_k, x) &= 0. \end{aligned}\right\} \quad x > 0. \quad (4)$$

Solutions of the third kind. (Temperature step at one wall; the opposite wall adiabatic):

$$\left.\begin{aligned} T(r_j, x) &= T_w \\ T_r(r_k, x) &= 0 \end{aligned}\right\} \quad x > 0. \quad (5)$$

Solutions of the fourth kind. (Step in heat flux at one wall; the opposite wall kept at the inlet temperature):

$$\left.\begin{aligned} T_r(r_j, x) &= sq_j''/k \\ T(r_k, x) &= T_e \end{aligned}\right\} \quad x > 0. \quad (6)$$

The following dimensionless parameters and variables are defined:

$$\bar{r} = \frac{r}{r_o}$$

$$r^* = \frac{r_i}{r_o}$$

$$R = s\frac{1 - r^*}{\bar{r}_j}$$

$$\bar{x} = \frac{4x/D_h}{Pe}$$

$$\bar{y} = s\frac{\bar{r} - \bar{r}_j}{1 - r^*}$$

$$\bar{u} = \frac{u}{u_m}$$

$$\bar{T} = \frac{T - T_e}{T_w - T_e}$$

for solutions of the first and the third kind;

$$\bar{T} = \frac{T - T_e}{(r_o - r_i)\, q_j''/k}$$

for solutions of the second and the fourth kind.

For the circular tube one has

$$r^* = 0; \qquad j = o; \qquad s = -1;$$

and for the flat duct

$$r^* = 1; \qquad j = i; \qquad s = +1.$$

When the new variables are introduced in equation (1) one obtains

$$\bar{u}\bar{T}_{\bar{x}} = \bar{T}_{\bar{y}\bar{y}} + \frac{R}{1 + R\bar{y}}\bar{T}_{\bar{y}}. \quad (7)$$

For the annular duct the velocity is given by

$$\begin{aligned} \bar{u} &= \frac{2}{M}(1 - \bar{r}^2 + B \ln \bar{r}) \\ &= \frac{2r^{*2}}{M}[B \ln(1 + R\bar{y}) - 2R\bar{y} - R^2\bar{y}^2] \end{aligned} \quad (8)$$

where $B = (1 - r^{*2})/(-\ln r^*)$ and $M = 1 + r^{*2} - B$.

For $-1 < R\bar{y} < 1$ the expression (8) can be expanded in powers of $\bar{y}$,

$$\bar{u} = A\bar{y}(1 + \sum_{n=1}^{\infty} a_n \bar{y}^n) \tag{9}$$

where

$$A = \frac{2R}{M}(B - 2\bar{r}_j^2),$$

$$a_1 = -\frac{R(B + 2\bar{r}_j^2)}{2(B - 2\bar{r}_j^2)},$$

and

$$a_n = \frac{(-R)^n B}{(n+1)(B - 2\bar{r}_j^2)}, \qquad n = 2, 3, \ldots.$$

For the circular tube we have

$$A = 4,$$
$$a_1 = -\tfrac{1}{2},$$
$$a_n = 0, \qquad n = 2, 3, \ldots.$$

and for the flat duct

$$A = 6,$$
$$a_1 = -1,$$
$$a_n = 0, \qquad n = 2, 3, \ldots.$$

We now introduce the similarity variable from the Lévêque solution,

$$\eta = \bar{y}/\xi,$$

where

$$\xi = (9\bar{x}/A)^{\frac{1}{3}},$$

and a dimensionless temperature function $\theta(\xi, \eta)$,

$\theta = \bar{T}$ for solutions of the third kind,

and

$\theta = \bar{T}/\xi$ for solutions of the second kind.

For sufficiently small values of ξ a solution of the form

$$\theta(\xi, \eta) = \sum_{n=0}^{\infty} \xi^n \theta_n(\eta). \tag{10}$$

is assumed. Introducing this expression, together with the expansion of the velocity profile (9), into equation (7), letting $\bar{y} = \xi\eta$, and equating like powers of ξ, we obtain the following system of ordinary differential equations,

$$\theta_o'' + 3\eta^2\theta_o' - 3p\eta\theta_o = 0 \tag{11a}$$

$$\begin{aligned}\theta_n'' + 3\eta^2\theta_n' - 3(n+p)\eta\theta_n &= \sum_{m=1}^{n} \{[(-R)^m - 3a_m\eta^3]\,\eta^{m-1}\,\theta_{n-m}' \\ &\quad + 3(n - m + p)\,a_m\eta^{m+1}\,\theta_{n-m}\}\end{aligned} \tag{11b}$$

$n = 1, 2, \ldots$

$$p = \begin{cases} 0 \text{ for solutions of the third kind.} \\ 1 \text{ for solutions of the second kind.}\end{cases}$$

For solutions of the second kind the boundary conditions are

$$\theta_o'(0) = -1; \qquad \theta_o(\infty) = 0 \tag{12a}$$

$$\theta_n'(0) = 0; \qquad \theta_n(\infty) = 0 \tag{12b}$$

and for solutions of the third kind we have

$$\theta_o(0) = 1; \qquad \theta_o(\infty) = 0 \tag{13a}$$

$$\theta_n(0) = 0; \qquad \theta_n(\infty) = 0. \tag{13b}$$

The second of the boundary conditions in (12a) and (13a) follows from the requirement of the similarity solution that two of the three original boundary conditions must combine into a single condition. At first sight this may seem to indicate that it is solutions of the first and the fourth kind we obtain. However, an examination of the asymptotic behavior of the solutions shows that for all n both θ_n and θ_n' tend to zero for large values of η. Although, in principle, this corresponds to an adiabatic wall at infinity, the actual behavior of the solutions is such that θ_n' is of the order of 10^{-5}–10^{-6} already for values of η between 2·5 and 3. Thus, as long as

$$\xi\eta_{\max} \leqslant 1,$$

where $\eta_{\max}$ is the value for which θ_n' is, in effect, zero, we have the adiabatic condition at the opposite wall. The aforementioned condition consequently determines the maximum value of $\bar{x}$ for which the solutions will be valid,

$$\bar{x}_{max} = A/(9\eta^3{}_{max}).$$

For annular ducts with the outer wall heated the values of $\bar{x}_{max}$ vary between those for the circular tube and the flat duct, respectively. With $\eta_{max} = 2{\cdot}5$ this range is

$$0{\cdot}028 \leqslant \bar{x}_{max} \leqslant 0{\cdot}042.$$

Values of $\bar{x}_{max}$ for annuli with the inner wall heated and $r^* \geqslant 0{\cdot}5$ lie above the value for the flat duct; if, however, the radius ratio is less than 0·5, a more restrictive condition follows from the power series expansion of the velocity profile, namely $R\bar{y} < 1$. This gives

$$\bar{x}_{max} = \frac{r^*}{1 - r^*} \frac{A}{9\eta_{max}^3}.$$

For a step-change in the wall temperature ($p = 0$) the solution of the unperturbed equation (11a) is the well-known Lévêque solution,

$$\theta_o^{(3)} = 1 - \frac{1}{\Gamma(\frac{4}{3})} \int_0^\eta e^{-\mu^3}\, d\mu. \tag{14}$$

The corresponding solution for uniform heat flux ($p = 1$) is

$$\theta_o^{(2)} = \frac{2}{3\,\Gamma(\frac{5}{3})} e^{-\eta^3} - \eta\left[1 - \frac{2}{\Gamma(\frac{5}{3})} \int_0^\eta \mu e^{-\mu^3}\, d\mu\right]. \tag{15}$$

Of the higher order equations only the one of first order for a step-change in the wall temperature has a simple solution,

$$\theta_1^{(3)} = -\left(\frac{R}{2} + \frac{a_1}{5}\right)\eta\theta_o^{(3)} + \frac{a_1}{5}\eta^2\theta_o^{(3)\prime} \tag{16}$$

The remaining equations, up to order six, were solved numerically on the IBM 7090 computer of Northern Europe University Computing Center, Lyngby, Denmark. A direct finite-difference method was used with difference corrections including those of sixth order.

From equations (15) and (14), respectively, one finds

$$\theta_o^{(2)}(0) = 0{\cdot}73849;$$

$$\theta_o^{(3)\prime}(0) = -1{\cdot}11984.$$

The values of $\theta_n^{(2)}(0)$ and $\theta^{(3)\prime}(0)$ for the higher-order solutions are given in Tables 1–4.

Table 1. Solutions of the second kind. Step in heat flux at the inner wall

r^*	$\theta_{i1}^{(2)}(0)$	$\theta_{i2}^{(2)}(0)$	$\theta_{i3}^{(2)}(0)$	$\theta_{i4}^{(2)}(0)$	$\theta_{i5}^{(2)}(0)$	$\theta_{i6}^{(2)}(0)$
0·10	−1·69193	5·13625	−17·2187	62·548	−243·58	1017·9
0·25	−0·51254	0·53520	−0·55792	0·67666	−0·83333	1·14761
0·5	−0·10992	0·06597	−0·00056	0·01662	0·00779	0·01027
1·0	0·09635	0·04004	0·02292	0·01575	0·01232	0·01068

Table 2. Solutions of the second kind. Step in heat flux at the outer wall

r^*	$\theta_{o1}^{(2)}(0)$	$\theta_{o2}^{(2)}(0)$	$\theta_{o3}^{(2)}(0)$	$\theta_{o5}^{(2)}(0)$	$\theta_{o5}^{(2)}(0)$	$\theta_{o6}^{(2)}(0)$
0	0·28904	0·13816	0·07174	0·03957	0·02317	0·01447
0·02	0·29709	0·14945	0·08297	0·04951	0·03164	0·02171
0·05	0·29287	0·14655	0·08146	0·04894	0·03163	0·02203
0·10	0·28388	0·13947	0·07684	0·04613	0·03001	0·02116
0·25	0·25369	0·11615	0·06156	0·03652	0·02401	0·01742
0·5	0·20130	0·08183	0·04165	0·02506	0·01729	0·01341

Table 3. Solutions of the third kind. Temperature step at the inner wall

r^*	$\theta_{i1}^{(3)\prime}(0)$	$\theta_{i2}^{(3)\prime}(0)$	$\theta_{i3}^{(3)\prime}(0)$	$\theta_{i4}^{(3)\prime}(0)$	$\theta_{i5}^{(3)\prime}(0)$	$\theta_{i6}^{(3)\prime}(0)$
0·10	−3·51217	4·11635	−12·4886	52·471	−248·11	1289·4
0·25	−1·06355	0·54222	−0·35923	0·64884	−0·82619	1·56288
0·5	−0·22818	0·13144	0·04316	0·05141	0·04067	0·04568
1·0	−0·20000	0·07357	0·04645	0·03639	0·03308	0·03272

Table 4. Solutions of the third kind. Temperature step at the outer wall

r^*	$\theta_{o1}^{(3)\prime}(0)$	$\theta_{o2}^{(3)\prime}(0)$	$\theta_{o3}^{(3)\prime}(0)$	$\theta_{o4}^{(3)\prime}(0)$	$\theta_{o5}^{(3)\prime}(0)$	$\theta_{o6}^{(3)\prime}(0)$
0	0·60000	0·08992	0·04313	0·02697	0·01943	0·01539
0·02	0·61671	0·10398	0·05497	0·03770	0·02971	0·02569
0·05	0·60795	0·10444	0·05597	0·03892	0·03111	0·02731
0·10	0·58929	0·10269	0·05551	0·03898	0·03150	0·02799
0·25	0·52663	0·09427	0·05158	0·03683	0·03040	0·02770
0·5	0·41787	0·08176	0·04660	0·03462	0·02972	0·02814

For solutions of the second kind the non-dimensional wall temperature is, to the Nth approximation,

$$\bar{T}_j^{(2)} = \sum_{n=0}^{N} \left(\frac{9\bar{x}}{A}\right)^{(n+1)/3} \theta_{j,n}^{(2)}(0). \tag{17}$$

Correspondingly, the non-dimensional heat flux for solutions of the third kind is

$$\Phi_j^{(3)} = -2 \sum_{n=0}^{N} \left(\frac{9x}{A}\right)^{(n-1)/3} \theta_{j,n}^{(3)\prime}(0). \tag{18}$$

The mixed mean temperatures, determined from an energy balance over the duct length $\bar{x}$, are

$$\bar{T}_{m,j}^{(2)} = \frac{2\bar{r}_j}{1+r^*}\,\bar{x} \tag{19}$$

$$\bar{T}_{m,j}^{(3)} = -\frac{2A}{3}\frac{\bar{r}_j}{1+r^*} \times \sum_{n=0}^{N} \frac{1}{n+2}\left(\frac{9\bar{x}}{A}\right)^{(n+2)/3} \theta_{j,n}^{(3)\prime}(0). \tag{20}$$

Finally, the Nusselt numbers become

$$Nu_j^{(2)} = 2/(\bar{T}_j - \bar{T}_{m,j}^{(2)}) \tag{21}$$

$$Nu_j^{(3)} = \Phi_j^{(3)}/(1 - \bar{T}_{m,j}^{(3)}). \tag{22}$$

In Tables 5–8 these quantities (with the exception of $\bar{T}_{m,j}^{(2)}$) are presented for selected values of $\bar{x}$.

From the nature of the present solution it is obvious that the results will be more accurate the smaller $\bar{x}$ is. As $\bar{x}$ increases the influence of the higher-order solutions becomes more pronounced and, due to the recursive method of solution, both rounding errors and truncation errors tend to build up. However, except for the cases mentioned below, the higher-order solutions are well-behaved in the sense that neither the functions themselves nor their derivatives increase significantly in magnitude with increasing order. In the finite-difference solution the iteration was continued until the maximum deviation between two successive sets of difference corrections was less than $5 \,.\, 10^{-6}$. The convergence was very rapid (with a step-width of 0·1 only two iterations were necessary), so the magnitude of the neglected difference corrections is probably not larger than 10^{-6}.

A less satisfactory behavior is found for the annular duct with the inner wall heated when r^* is less than 0·5. As r^* decreases below this value the magnitude of the higher order functions increase rapidly, and the convergence of the series solution becomes increasingly slow.

Table 5. Solutions of the second kind. Step in heat flux at the inner wall

r^*	$\bar{x}$	$\bar{T}_i^{(2)}$	$Nu_i^{(2)}$	Reference [11]
0·10	10^{-4}	0·02797	78·56	
	2.10^{-4}	0·03451	58·02	
	4.10^{-4}	0·04239	47·27	48·3
	10^{-3}	0·05519	36·37	36·7
0·25	10^{-4}	0·033762	59·308	
	2.10^{-4}	0·042203	47·480	
	4.10^{-4}	0·052654	38·100	39·5
	10^{-3}	0·070294	28·615	29·1
	2.10^{-3}	0·087183	23·153	23·3
	4.10^{-3}	0·10777	18·838	18·9
	10^{-2}	0·14179	14·52	14·52
	2.10^{-2}	0·1736	12·08	12·08
0·5	10^{-4}	0·037125	53·969	
	2.10^{-4}	0·046689	42·960	
	4.10^{-4}	0·058690	34·233	34·6
	10^{-3}	0·079350	25·418	25·6
	2.10^{-3}	0·099610	20·351	20·4
	4.10^{-3}	0·12495	16·356	16·4
	10^{-2}	0·16838	12·367	12·37
	2.10^{-2}	0·21082	10·127	10·13
	4.10^{-2}	0·26382	8·434	8·433
1	10^{-4}	0·039516	50·740	
	2.10^{-4}	0·049881	40·257	
	4.10^{-4}	0·062997	31·950	
	10^{-3}	0·085863	23·568	23·5
	2.10^{-3}	0·10864	18·754	
	4.10^{-3}	0·13764	14·965	
	10^{-2}	0·18869	11·193	11·2
	2.10^{-2}	0·24024	9·081	
	4.10^{-2}	0·30702	7·490	7·4895

Table 6. Solutions of the second kind. Step in heat flux at the outer wall

r^*	$\bar{x}$	$\bar{T}_o^{(2)}$	$Nu_o^{(2)}$	Reference [3]
0	10^{-4}	0·046018	43·651	
	2.10^{-4}	0·058358	34·511	
	4.10^{-4}	0·074126	27·276	
	10^{-3}	0·10207	19·987	
	2.10^{-3}	0·13048	15·813	(15·07)
	4.10^{-3}	0·16751	12·538	(12·42)
	10^{-2}	0·23517	9·295	9·293
	2.10^{-2}	0·30689	7·494	7·494
	4.10^{-2}	0·40528	6·149	6·148

Table 6—continued

r^*	$\bar{x}$	$\bar{T}_o^{(2)}$	$Nu_o^{(2)}$	Reference [11]
0·02	10^{-4}	0·043935	45·726	
	2.10^{-4}	0·055706	36·157	
	4.10^{-4}	0·070753	28·584	29·3
	10^{-3}	0·097399	20·956	21·3
	2.10^{-3}	0·12448	16·589	16·8
	4.10^{-3}	0·15977	13·164	13·2
	10^{-2}	0·22419	9·776	9·784
	2.10^{-2}	0·22243	7·898	7·898
	4.10^{-2}	0·38601	6·502	6·5017
0·05	10^{-4}	0·043471	46·210	
	2.10^{-4}	0·055109	36·544	
	4.10^{-4}	0·069981	28·894	29·7
	10^{-3}	0·096302	21·187	21·5
	2.10^{-3}	0·12304	16·774	16·9
	4.10^{-3}	0·15784	13·314	13·4
	10^{-2}	0·22130	9·889	9·895
	2.10^{-2}	0·28840	7·990	7·991
	4.10^{-2}	0·38022	6·578	6·5776
0·10	10^{-4}	0·043006	46·703	
	2.10^{-4}	0·054505	36·940	
	4.10^{-4}	0·061192	29·212	29·9
	10^{-3}	0·095162	21·426	21·7
	2.10^{-3}	0·12151	16·967	17·1
	4.10^{-3}	0·15576	13·469	13·50
	10^{-2}	0·21808	10·005	10·01
	2.10^{-2}	0·28379	8·083	8·083
	4.10^{-2}	0·37337	6·652	6·6517
0·25	10^{-4}	0·042127	47·657	
	2.10^{-4}	0·053353	37·713	
	4.10^{-4}	0·067665	29·840	30·2
	10^{-3}	0·092907	21·904	22·0
	2.10^{-3}	0·11843	17·356	17·4
	4.10^{-3}	0·15148	13·786	13·80
	10^{-2}	0·21122	10·245	10·25
	2.10^{-2}	0·27369	8·275	8·275
	4.10^{-2}	0·35799	6·803	6·8025
0·5	10^{-4}	0·041098	48·823	
	2.10^{-4}	0·051989	38·668	
	4.10^{-4}	0·065837	30·626	30·8
	10^{-3}	0·090161	22·516	22·6
	2.10^{-3}	0·11463	17·864	17·9
	4.10^{-3}	0·14611	14·207	14·21
	10^{-2}	0·20247	10·574	10·57
	2.10^{-2}	0·26065	8·548	8·547
	4.10^{-2}	0·33795	7·027	7·0265

Since, furthermore, the range of $\bar{x}$ in which the solutions are valid decreases as $r^*/(1 - r^*)$, solutions with heating at the inner wall are given only for $r^* \geqslant 0·1$.

Table 7. Solutions of the third kind. Temperature step at the inner wall

r^*	$\bar{x}$	$\Phi_i^{(3)}$	$\bar{T}_{im}^{(3)}$	$Nu_i^{(3)}$	Reference [11]
0·10	10^{-4}	60·873	0·00080	60·922	
	2.10^{-4}	49·632	0·00129	49·696	
	4.10^{-4}	40·68	0·00210	40·77	40·4
	10^{-3}	31·57	0·00402	31·70	31·5
0·25	10^{-4}	49·538	0·00147	49·611	
	2.10^{-4}	39·734	0·00234	39·828	
	4.10^{-4}	31·948	0·00375	32·068	32·0
	10^{-3}	24·053	0·00702	24·223	24·2
	2.10^{-3}	19·483	0·01130	19·705	19·7
	4.10^{-3}	15·844	0·01826	16·139	16·11
	10^{-2}	12·138	0·03460	12·574	12·56
	2.10^{-2}	9·98	0·0564	10·57	10·57
0·5	10^{-4}	44·669	0·00223	44·768	
	2.10^{-4}	35·541	0·00354	35·668	
	4.10^{-4}	28·295	0·00563	28·456	28·4
	10^{-3}	20·954	0·01041	21·175	21·1
	2.10^{-3}	16·711	0·01658	16·993	17·0
	4.10^{-3}	13·339	0·02642	13·701	13·68
	10^{-2}	9·916	0·04900	10·427	10·42
	2.10^{-2}	7·930	0·07824	8·603	8·602
	4.10^{-2}	6·341	0·12501	7·247	7·2465
1	10^{-4}	41·744	0·00314	41·876	
	2.10^{-4}	33·046	0·00498	33·212	
	4.10^{-4}	26·141	0·00788	26·349	
	10^{-3}	19·147	0·01447	19·428	19·4
	2.10^{-3}	15·106	0·02288	15·459	
	4.10^{-3}	11·895	0·03613	12·341	
	10^{-2}	8·638	0·06596	9·248	9·25
	2.10^{-2}	6·750	0·10371	7·531	
	4.10^{-2}	5·243	0·16253	6·260	6·259

Table 8. Solutions of the third kind. Temperature step at the outer wall

r^*	$\bar{x}$	$\Phi_o^{(3)}$	$\bar{T}_{om}^{(3)}$	$Nu_o^{(3)}$	Reference [12]
0	10^{-4}	35·612	0·00540	35·806	35·80
	2.10^{-4}	28·013	0·00853	28·254	28·26
	4.10^{-4}	21·979	0·01343	22·278	22·29
	10^{-3}	15·867	0·02442	16·264	16·28
	2.10^{-3}	12·334	0·03825	12·824	12·82
	4.10^{-3}	9·526	0·05968	10·130	10·13
	10^{-2}	6·674	0·10657	7·470	7·470
	2.10^{-2}	5·019	0·16378	6·002	6·002
	4.10^{-2}	3·693	0·24890	4·917	4·916
0·02	10^{-4}	37·306	0·00555	37·515	Reference [11]
	2.10^{-4}	29·350	0·00876	29·609	
	4.10^{-4}	23·033	0·01380	23·355	23·5
	10^{-3}	16·633	0·02509	17·061	17·2
	2.10^{-3}	12·934	0·03930	13·463	13·6
	4.10^{-3}	9·993	0·06134	10·646	10·8

Table 8—continued

r^*	$\bar{x}$	$\Phi_o^{(3)}$	$\bar{T}_{om}^{(3)}$	$Nu_o^{(3)}$	Reference [11]
	10^{-2}	7·006	0·10958	7·869	7·914
	2.10^{-2}	5·272	0·16848	6·340	6·350
	4.10^{-2}	3·881	0·25616	5·218	5·2174
0·05	10^{-4}	37·714	0·00545	37·920	
	2.10^{-4}	29·676	0·00860	29·934	
	4.10^{-4}	23·296	0·01355	23·616	23·7
	10^{-3}	16·832	0·02464	17·257	17·4
	2.10^{-3}	13·095	0·03862	13·621	13·7
	4.10^{-3}	10·125	0·06031	10·774	10·9
	10^{-2}	7·108	0·10782	7·967	8·000
	2.10^{-2}	5·356	0·16591	6·422	6·428
	4.10^{-2}	3·952	0·25255	5·287	5·2868
0·10	10^{-4}	38·135	0·00525	38·367	
	2.10^{-4}	30·019	0·00830	30·270	
	4.10^{-4}	23·576	0·01308	23·888	24·0
	10^{-3}	17·048	0·02380	17·464	17·5
	2.10^{-3}	13·274	0·03732	13·789	13·9
	4.10^{-3}	10·276	0·05832	10·912	11·00
	10^{-2}	7·230	0·10440	8·073	8·095
	2.10^{-2}	5·462	0·16087	6·508	6·513
	4.10^{-2}	4·045	0·24535	5·360	5·3590
0·25	10^{-4}	38·972	0·00472	39·157	
	2.10^{-4}	30·710	0·00746	30·940	
	4.10^{-4}	24·150	0·01176	24·438	24·5
	10^{-3}	17·506	0·02144	17·889	17·9
	2.10^{-3}	13·665	0·03368	14·141	14·2
	4.10^{-3}	10·613	0·05273	11·204	11·2
	10^{-2}	7·515	0·09475	8·301	8·311
	2.10^{-2}	5·717	0·14658	6·698	6·700
	4.10^{-2}	4·278	0·22480	5·518	5·5176
0·5	10^{-4}	40·015	0·00403	40·177	
	2.10^{-4}	31·583	0·00637	31·786	
	4.10^{-4}	24·890	0·01007	25·143	25·1
	10^{-3}	18·109	0·01840	18·448	18·5
	2.10^{-3}	14·191	0·02897	14·614	14·6
	4.10^{-3}	11·077	0·04549	11·605	11·6
	10^{-2}	7·918	0·08222	8·628	8·629
	2.10^{-2}	6·086	0·12798	6·979	6·980
	4.10^{-2}	4·622	0·19789	5·762	5·7615

COMPARISON WITH PREVIOUS SOLUTIONS

In Tables 5–8 Nusselt numbers from other sources are included for comparison with the present results. For the annular passages and the flat duct the solution by Lundberg *et al.* [7] is taken.† The values for the circular tube with a step-change in wall temperature are those presented by Munakata [12] who used the eigenvalues and eigenfunctions computed by Brown [1]. For the circular tube with a step-change in heat flux at the wall Nusselt numbers were computed from the eigenvalues and related constants given by Sellars *et al.* [3].

In the downstream part of the range the agreement with the eigenvalue solutions is

† Numerical values are taken from reference [11].

excellent. For small values of $\bar{x}$, however, and in particular where the previous results are based on the simple Lévêque solution, the present solution yields more correct values. The results for the circular tube with a step-change in the wall heat-flux show that the seven eigenvalues computed by Sellars *et al.* are sufficient only for $\bar{x} > 10^{-2}$. This indicates that from the point of view of computational effort it would be advantageous to switch from the eigenvalue solution to the extended Lévêque solution at values of $\bar{x}$ in the neighborhood of 10^{-2}.

For the circular tube with a step-change in the wall temperature Munakata [12] obtained an improved asymptotic solution for small values of $\bar{x}$ by replacing the summation of eigenfunctions by the corresponding integral. It is interesting to note that the resulting expressions are essentially the same as those obtained from the first order perturbation of the Lévêque solution.†

CONCLUDING REMARKS

The present analysis and the subsequent numerical results have clearly demonstrated the usefulness of the extended Lévêque solution for computing heat-transfer rates or wall temperatures in the thermal entrance region with fully developed laminar flow. With a maximum of six terms in the fundamental solutions this method takes over at the point where the eigenvalue solutions become cumbersome due to the large number of terms required. The fact that with this type of solution one cannot discriminate between different kinds of boundary conditions at the opposite wall is of no consequence if the solutions are used only for $\bar{x} \leqslant 10^{-2}$; up to this point the differences between solutions of the first and the third kind and between solutions of the second and the fourth kind are quite negligible.

† For the heat flux at the wall Munakata's result is (in terms of the present notation):

$$\Phi = 1{\cdot}3566(x/4)^{-\frac{1}{3}} - 1{\cdot}21$$

Apart from a minor difference in the constant term (1·21 vs. 6/5) this is identical with equation (18) for $N = 1$ and $r^* = 0$.

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Résumé—Des solutions en séries du type proposé pour la première fois par Mercer [9, 10] sont présentées pour l'écoulement laminaire dans des tubes et des conduites annulaires (y compris la conduite à section en rectangle aplati) avec une variation en échelon soit de la température soit du flux de chaleur à l'une des parois. Ce type de solution—qui peut être regardé comme une extension de la solution bien connue de

Lévêque [8] constitue une alternative convenable aux solutions avec valeurs propres dans la première partie des régions d'entrée thermique où un grand nombre de termes sont nécessaires dans les développements en fonctions propres.

Zusammenfassung—Reihenlösungen von der Art, wie sie zuerst von Mercer [9, 10] vorgeschlagen wurden, sind für laminare Strömung in Rohren und Ringspalten (einschliesslich ebener Kanäle) mit einer sprunghaften Änderung entweder der Temperatur oder des Wärmeflusses an einer Wand angegeben. Diese Art der Lösung—sie kann als Erweiterung der Leveque–Lösung [8] gelten—stellt eine bequeme Alternative zur Eigenwertlösung für den ersten Teil des thermischen Einlaufbereichs dar, für den eine grosse Zahl von Ausdrücken für die Eigenfunktionsentwicklung erforderlich sind.

Аннотация—Для ламинарного течения в трубах и кольцевых каналах, включая плоские каналы с изменяющейся ступенчатой температурой или плотностью теплового потока, получены решения в виде рядов типа, впервые предложенного Мерсером [9, 10]. Этот тип решения, который можно рассматривать как дальнейшее развитие известного решения Левека [8], удобно использовать в начале теплового входного участка, где для разложений по собственным функциям требуется большое количество членов.